Math Day 2024
at Murray State University
Upper Level Examination

- Do not open this exam until you are told to do so.
- Clearly fill in your NAME and STUDENT NUMBER on the bubble sheet. Your student number is located on the card your teacher gave you.
- You have 50 minutes to complete this exam.
- You may not use a calculator, phone, notes, book, or other aid. Any attempt to do so will result in disqualification.
- The exam will be scored as follows:

  $+1$ point for a correct answer
  $-\frac{1}{4}$ point for an incorrect answer
  0 points for a blank answer

- Clearly select one answer on the bubble sheet for each question. If more than one answer is selected, the answer will be marked as incorrect.

GOOD LUCK!

1. Suppose that S is a subset of natural numbers $\{1, 2, 3, 4, ...\}$. Which set of conditions guarantees that $S = \{1, 2, 3, 4, ...\}$?

   (a) $n \in S$ implies $n + 1 \in S$ is true.

   (b) $1 \in S$ is true, $2 \in S$ is true, and $n \in S$ implies $n + 2 \in S$ is true.

   (c) $1 \in S$ is true, $3 \in S$ is true, and $n \in S$ implies $n + 2 \in S$ is true.

   (d) $1 \in S$ is true, $2 \in S$ is true, and $n \in S$ implies $n + 3 \in S$ is true.

   (e) $1 \in S$ is true and $n \in S$ implies $n - 1 \in S$ is true.

2. Let R be the region in the 1st quadrant bounded by $y(t) = \sqrt{4 - t^2}$, $t = 0$, and $y = 0$. Find the area of R.

   (a) $\frac{\pi}{2}$

   (b) π

   (c) 2π

   (d) 4π

   (e) None of the above

3. Consider the following statement:

   If condition P is true, then at most one of conditions Q or R is true.

   Which of the following statements is equivalent to the above statement?

   (a) If condition P is true and condition Q is false, then condition R is true.

   (b) If condition P is true and condition Q is false, then condition R is false.

   (c) If condition P is false and condition Q is false, then condition R is true.

   (d) If condition P is false and condition Q is false, then condition R is false.

   (e) None of the above

4. Suppose there is a bag containing 2 yellow marbles, 2 red marbles, and 2 orange marbles. If two balls are drawn randomly (without replacement of the 1st ball), what is the probability of drawing matching color marbles?

   (a) $\frac{1}{18}$
   (b) $\frac{1}{15}$
   (c) $\frac{1}{6}$
   (d) $\frac{1}{5}$
   (e) $\frac{1}{3}$

5. Suppose that
$$S(n) = 1 + 3 + 5 + ... + (2n - 1).$$
   Evaluate $S(1000)$.

   (a) $S(1000) = 500500$
   (b) $S(1000) = 1000000$
   (c) $S(1000) = 1000500$
   (d) $S(1000) = 2000000$
   (e) None of the above

6. Evaluate $\csc^2\left(\frac{\pi}{8}\right)$.

   (a) $\frac{2\sqrt{2}}{\sqrt{2}-1}$
   (b) $\frac{2\sqrt{2}}{1-\sqrt{2}}$
   (c) $\frac{2}{\sqrt{2}+1}$
   (d) $\frac{2}{1-\sqrt{2}}$
   (e) $\frac{\sqrt{2}}{\sqrt{2}+1}$

7. Suppose that a given exam for a disease tests accurately 95% of the time. Also suppose the population in a town has a 1% infection rate. To the nearest percent, what is the approximate probability that a person living in the town who tests positive is actually infected?

   (a) 1%
   (b) 5%
   (c) 16%
   (d) 95%
   (e) 99%

8. The solution x_0 for the equation $(\log_2(x))^2 + 1 = 2\log_2(x)$ satisfies

   (a) $0 < x_0 \leq 1$
   (b) $1 < x_0 \leq 2$
   (c) $2 < x_0 \leq 3$
   (d) $3 < x_0 \leq 4$
   (e) None of the above

9. Evaluate
$$\sin^2\left(\cos^{-1}(0.8)\right).$$

   (a) 0.125
   (b) 0.36
   (c) 0.6
   (d) 0.64
   (e) None of the above

10. Calculate the following sum when $n \geq 2$:
$$1 + 2 + 3 + 4 + ... + 2^n.$$

   (a) 2^{2n-1}
   (b) 4^n
   (c) $2^{2n-1} + 2^{n-2}$
   (d) $4^n + 2^{n-2}$
   (e) None of the above

11. Suppose $f(x) = \sin(x)$ on the domain $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$, and define $g(x)$ to be its inverse function. Evaluate

$$g \quad \cos \ \frac{\pi}{6} \qquad .$$

(a) $\dfrac{\pi}{3}$

(b) $\dfrac{2\pi}{3}$

(c) $\dfrac{5\pi}{6}$

(d) $\dfrac{7\pi}{6}$

(e) $\dfrac{4\pi}{3}$

12. Find the equation of the line that is tangent to $f(x) = \sqrt{x^2 + 8}$ at $x = 1$.

(a) $y = \dfrac{1}{6}(x - 1) + 3$

(b) $y = \frac{1}{3}(x - 1) + 3$

(c) $y = \dfrac{2}{3}(x - 1) + 3$

(d) $y = \sqrt{2}(x - 1) + 3$

(e) None of the above

13. If s_1 and s_2 are the sample standard deviations for the sets $\{1, 5, 9\}$ and $\{10, 13, 16\}$, respectively, then we have

(a) $s_1 > s_2 \geq 4$

(b) $4 \geq s_1 > s_2$

(c) $s_2 > s_1 \geq 4$

(d) $4 \geq s_2 > s_1$

(e) None of the above

14. Suppose both the following statements are true:

   I. If it is cold and rainy, then Jill is sad.
   II. If it is Friday, then Jill is not sad.

   Suppose Jill is sad. What can be concluded?

   (a) It is Friday and it is both rainy and cold.
   (b) It is Friday and it is both not rainy and not cold.
   (c) It is not Friday and it is either rainy or cold.
   (d) It is not Friday and it is both rainy and cold.
   (e) None of the above

15. Suppose exam scores follow a normal distribution with a mean of 75% and standard deviation of 7%. What is the minimum score that was better than 99.85% of all test takers?

   (a) 96%
   (b) 97%
   (c) 98%
   (d) 99%
   (e) 100%

16. Suppose you are standing on top of a tower and sight a car with angle of depression 60°. If the base of the tower is 100 feet away from the car, how far away are you from the car?

   (a) $\frac{200}{\sqrt{2}}$ feet
   (b) $\frac{200}{\sqrt{3}}$ feet
   (c) 150 feet
   (d) 200 feet
   (e) None of the above

17. Evaluate $\lim_{x \to 3^+} \dfrac{|x^2 - 7x + 12|}{x - 3}$, if the limit exists.

   (a) 0
   (b) -7
   (c) 1
   (d) -1
   (e) The limit does not exist.

18. Assume you roll a 20 inch diameter ball along a line with an angular velocity of $360 \, \dfrac{\text{degrees}}{\text{second}}$. How far will the ball roll in 10 seconds?

   (a) 100π inches
   (b) 200π inches
   (c) 400π inches
   (d) 1000π inches
   (e) None of the above

19. Three men are told to stand in a straight line, one in front of the other. A hat is put on each of their heads. They are told that each of these hats was selected from a group of five hats: two black hats and three white hats. The first man, standing at the front of the line, can't see either of the men behind him or their hats. The second man, in the middle, can see only the first man and his hat. The last man, at the rear, can see both other men and their hats. The last man and middle man are asked in succession if they can deduce the color of his own hat to which both cannot. What must be true?

   (a) The first man has enough information to know that his hat is black.
   (b) The first man has enough information to know that his hat is white.
   (c) The first man would know the color of his hat only if he knew the color of the last man's hat.
   (d) The first man would know the color of his hat only if he knew the color of the middle man's hat.
   (e) The first man does not have sufficient information to know the color of his hat.

20. Characterize the end behavior for $f(t) = (-2t+5)^{100}(t^2+t+7)^{13}$.

(a) As $t \to \infty$, $f(t) \to \infty$ and as $t \to -\infty$, $f(t) \to \infty$.

(b) As $t \to \infty$, $f(t) \to \infty$ and as $t \to -\infty$, $f(t) \to -\infty$.

(c) As $t \to \infty$, $f(t) \to -\infty$ and as $t \to -\infty$, $f(t) \to \infty$.

(d) As $t \to \infty$, $f(t) \to -\infty$ and as $t \to -\infty$, $f(t) \to -\infty$.

(e) None of the above

21. Suppose a property is defined as follows:

For all x, there exists y such that $z < x$ implies $f(z) < y$.

What does it mean for this property NOT to hold?

(a) There exists x such that for all y and some $z < x$, we have $f(z) \geq y$.

(b) There exists x such that for all y and some $z \geq x$, we have $f(z) \geq y$.

(c) There exists x such that for all y, if $z < x$ then $f(z) \geq y$.

(d) There exists x such that for all y, if $z \geq x$ then $f(z) \geq y$.

(e) None of the above

22. Let n be a natural number and define $g(t) = \begin{cases} t^n, & \text{if } t > 0 \\ -t^n, & \text{if } t \leq 0. \end{cases}$

If $f(x) = \dfrac{d^n}{dx^n} g(x)$, calculate $f(0)$.

(a) $f(0) = n!$

(b) $f(0) = -n!$

(c) $f(0) = 0$

(d) $f(0)$ does not exist.

(e) None of the above

23. Consider the following data set:

$$1, 3, 3, 4, 5, 7, 7, 8, 8, 10, 12, 14, 20, 22, 45, 50, 60$$

What is the smallest datum point that lies in the 75th percentile?

(a) 20
(b) 22
(c) 45
(d) 50
(e) None of the above

24. How many solutions (in radians) exist for $2\pi \sin^2(x) = \pi - x$ in $0 \leq x \leq 2\pi$?

(a) 1
(b) 2
(c) 3
(d) 4
(e) None of the above

25. What is an equivalent expression for $2\sin(x)\cos(y)$?

(a) $\sin(x - y) + \cos(x + y)$
(b) $\sin(x - y) - \cos(x + y)$
(c) $\cos(x - y) + \cos(x + y)$
(d) $\cos(x - y) - \cos(x + y)$
(e) None of the above

26. Find x in the following pattern:

$$4, 11, 31, 89, 259, x$$

(a) 461
(b) 561
(c) 661
(d) 761
(e) None of the above

27. Suppose we have the relationship $y(x)^3 + y(x) = x$. Find the equation of the tangent line to $y(x)$ at the point $(2, 1)$.

   (a) $y = 1$
   (b) $y = \frac{1}{6}x + \frac{2}{3}$
   (c) $y = -2x + 5$
   (d) $y = \frac{1}{13}x + \frac{11}{13}$
   (e) None of the above

28. How many x-intercepts does the function $R(x) = \dfrac{x^3 + 7x^2 + 2x - 40}{x^2 - 5x + 6}$ have?

   (a) 1
   (b) 2
   (c) 3
   (d) 4
   (e) 5

29. Given that $a > 0$, $\sec\theta = -\sqrt{a^2+1}$ and that θ satisfies $\pi < \theta < \dfrac{3\pi}{2}$, evaluate $\tan\theta$.

   (a) a
   (b) $\frac{1}{a}$
   (c) $-a$
   (d) $-\frac{1}{a}$
   (e) None of the above

30. Let $y(x)$ be a differentiable function satisfying $y'(x) = 8x(x^2+1)^3$. If $y(0) = 1$, find the value of $y(2)$.

   (a) 625
   (b) 2000
   (c) 2500
   (d) 48^3
   (e) None of the above